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ASYMPTOTIC STABILITY OF NON-AUTONOMOUS UPPER TRIANGULAR SYSTEMS AND A GENERALIZATION OF LEVINSON'S THEOREM

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Abstract. This article studies asymptotic stability of non-autonomous linear systems with time-dependent coefficient matrices $\{A(t)\}_{t\in\mathbb{R}}$. The classical theorem of Levinson has been widely used to science and engineering non-autonomous systems, but systems with defective eigenvalues could not be covered because such a family does not allow continuous diagonalization. We study systems where the family allows to have upper triangulation and to have defective eigenvalues. In addition to the wider applicability, working with upper triangular matrices in place of Jordan form matrices offers more flexibility. We interpret our and earlier works including Levinson's theorem from the perspective of invariant manifold theory.

1. Introduction

This article is devoted to the study of asymptotic stability for

$$x'(t) = A(t)x(t), \tag{1.1}$$

with $x(t) \in \mathbb{R}^N$, $A(t) \in \mathbb{R}^{N\times N}$ for each $t \in \mathbb{R}$. In particular, we study when $A(t) = U(t) + \mathcal{E}(t)$ for $U(t)$ upper triangular and $\mathcal{E}(t)$ suitably small.

Our theorem (Theorem 3.2) generalizes the classical Levinson's theorem [9], which has been widely used to study asymptotic stability of

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non-autonomous systems. Levinson's theorem applies to a class of problems with coefficient suitably diagonalizable and time-dependent. The theorem can be found in many places [2, 4, 8], and for the readers convenience we include the Levinson theorem in the Appendix. The theorem has been applied to tremendously many science and engineering non-autonomous systems.

To illustrate our motivation for generalization, consider the following simple example,

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}' = \begin{pmatrix} -1+\frac{1}{t} & 1 & 0 \\ 0 & -1-\frac{1}{t} & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} \tag{1.2}$$

and let $U(t) := \begin{pmatrix} -1+\frac{1}{t} & 1 & 0 \\ 0 & -1-\frac{1}{t} & 0 \\ 0 & 0 & -2 \end{pmatrix}$. By integrating the system, we see that the trivial solution is asymptotically stable. One is asked if this stability can be persistently continued under suitable perturbations. Consider

$$x'(t) = U(t)x(t) + \mathcal{E}(t)x(t)$$

with $\mathcal{E}(t)$ a perturbation whose smallness will be stated in the sequel. For this example, the three eigenvalues of $U(t)$ are distinct for every finite t, hence $U(t)$ is (continuously) diagonalizable for every finite t. However, It fails to be so in the limit $t \to \infty$. Because of this failure at infinity, Levinson's theorem for this case is an empty statement; i.e., no result regarding asymptotic stability is provided by the theorem. As a matter of fact, the trivial solution is asymptotically stable with perturbations satisfying the smallness conditions for Levinson's theorem. Thus, there are grounds to generalize the theorem hypotheses to provide the same conclusions. In particular, to use upper triangular factorization in place of diagonalization is our main concern, allowing defective eigenvalues.

A few previous studies have considered this approach for systems with defective eigenvalues, appealing to block diagonalization. Alternate systems with a single Jordan block have been considered [7] and subsequently extended to systems comprising several Jordan blocks [6, 2]. Systems with multiple Jordan blocks have also been considered under weak dichotomy assumptions [3, 1]. Section 4 discusses how these previous studies relate to the current work, and we provide the invariant subspace point of view on those results.

This paper completes the following theorem by the author in [8, Theorem 1]. Its tweaked version as well as its proof is included in Appendix.

The notations $|x(t)|_\theta$ and $\|\cdot\|$ in the statement, the weighted length and the matrix norm respectively, are defined in Section 2.4.

THEOREM ([8]). Suppose that $U(t) = diag(U_0(t), U_1(t))$ with $U_0(t)$ and $U_1(t)$ upper triangular and of dimensions $N_0 \times N_0$ and $N_1 \times N_1$ respectively. Let $N = N_0 + N_1$. We make the following assumptions.

1. $\int_{a_0}^{\infty} \|\mathcal{E}(t)\|\, dt < \infty$ for some a_0.
2. There is a real-valued function θ and constants $\delta > 0$ and $A \in \mathbb{R}$ such that
   (a) for any $t_2 \geq t_1$ and any $\lambda_{0,i}(t)$ $i = 1, \cdots, N_0$ of eigenvalues of $U_0(t)$

$$\int_{t_1}^{t_2} Re\, \lambda_{0,i}(t) - \theta(t)\, dt \leq A - (N_0 - 1 + \delta)\log(1 + t_2 - t_1),$$

   (b) and for any $t_2 \geq t_1$ and any $\lambda_{1,i}(t)$ $i = 1, \cdots, N_1$ of eigenvalues of $U_1(t)$

$$\int_{t_1}^{t_2} \theta(t) - Re\, \lambda_{1,i}(t)\, dt \leq A - (N_1 - 1 + \delta)\log(1 + t_2 - t_1).$$

Then there is a constant a and an N_0-dimensional subspace E_0 of $\mathbb{R}^N$ such that $x(a) \in E_0$ implies that $\lim_{t\to\infty} |x(t)|_\theta = 0$.

In that exposition, a system with two blocks $U(t) = diag\big(U_0(t), U_1(t)\big)$ was considered, with spectral gap between them. We showed only the stability result, i.e., the persistent existence of one part with smaller eigenvalue (in real parts) against the other part, whereas Levinson's theorem states the persistent existence of an orbit with intermediate eigenvalue; Levinson's theorem proof resembles that for center manifold theory, rather than stable manifold theory. Our objective is to complete the approach [8, Theorem 1], considering block diagonal systems with a block that spectrally intervenes.

The remainder of this paper is organized as follows. Section 2 provides some key features for non-autonomous system asymptotic stability problems (readers already familiar with those aspects may prefer to go directly to Section 3). Section 3 presents the Theorem 3.2, the main outcome from the paper. Section 4 presents the invariant manifold point of view on this problem to clarify our aim to do and how this relates to earlier work.

2. Preliminaries

This section details our notations and introduce useful concepts from ordinary differential equation theories required to state the particular problem considered. Readers familiar with those basic notions may prefer to go directly to Section 3. Much of the detail in this section was from [8].

2.1. Key observations on asymptotic stability of non-autonomous systems

We use the following example from [10] to illustrate relevance of studying asymptotic stability of non-autonomous systems,

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \left\{ \begin{pmatrix} -\frac{1}{4} & 1 \\ -1 & -\frac{1}{4} \end{pmatrix} + \frac{3}{4} \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix} \right\} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Although the time-dependent coefficient matrix has eigenvalues $-\frac{1}{4} \pm \frac{\sqrt{7}}{4} i$ for all t, and hence the real part is always strictly negative, $\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$ $\exp(t/2)$ solves the equation. Thus, there is some deficiency in determining growth behavior from spectral information alone.

Another obvious but important observation is that the growth rate gap, derived from the spectral gap, is finer than that for constant coefficient systems. This can be illustrated by comparing systems

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}' = \begin{pmatrix} \frac{a}{t} & 0 \\ 0 & \frac{b}{t} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}' = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \quad a \neq b,$$

where the rate gap between independent solutions is t^{a-b} and $e^{(a-b)t}$, respectively. Consequently, adding perturbations of size $\mathcal{O}(\frac{1}{t})$ terms as $t \to \infty$ does spoil non-autonomous system asymptotic behavior. Levinson's theorem states that such a polynomial growth gap is kept under perturbations of integrable size.

2.2. Fundamental matrices

From the Picard-Lindelöf Theorem, a linear system has a unique solution for $|t - t_0| \leq \ell$ with $\ell = \min(a, b/M)$, where a, b, and M are such that in the domain $|t-t_0| \leq a$ and $|x-x_0| \leq b$, $f(t,x)$ is continuous in t and is uniformly Lipschitz in y and $|f(t,x)|$ is bounded by M.

For (1.1), we require
$(A0)$
1. Entries $\{A_{ij}(t)\}_{t\in\mathbb{R}}$, $i,j = 1,\cdots,N$ are continuous at all $t \in \mathbb{R}$.
2. $|A_{ij}(t)|$ is uniformly bounded by a constant $K > 0$.

We will call this assumption $(A0)$, and hence from the Picard-Lindelöf theorem the solution extends to all $\mathbb{R}$ uniquely (one could have considered a smooth cut-off if necessary).

Therefore, for any two numbers t and τ it makes senses to consider the solution matrix $\Phi(t,\tau)$, which maps $x(\tau)$ to $x(t)$ and the corresponding family $\{\Phi(t,\tau)\}_{t,\tau\in\mathbb{R}}$. The solution matrices for an autonomous system may be expressed as $\Phi(t-\tau)$, but they explicitly depend on t and τ for a non-autonomous system.

We have that $\Phi(t,t) = \mathbf{1}$ for all t and

$$\Phi(a,b)\Phi(b,c) = \Phi(a,c), \quad \forall a,b,c.$$

In particular, $\Phi(a,b)$ is always invertible with inverse $\Phi(b,a)$.

2.3. Operations on block diagonal matrices

Let $N_1, N_2, \cdots, N_k$ be fixed positive integers such that $\sum_1^k N_\alpha = N$. Consider a collection $\mathcal{C}$ of all block diagonal $N\times N$ matrices of the form $U = diag(B_1, B_2, \cdots, B_k)$, with blocks of dimensions $N_\alpha \times N_\alpha$. $\mathcal{C}$ is closed under the matrix multiplication. We find that for $U = diag(B_1, B_2, \cdots, B_k) \in \mathcal{C}$ and $W = diag(C_1, C_2, \cdots, C_k) \in \mathcal{C}$, $UW = diag(B_1C_1, B_2C_2, \cdots, B_kC_k) \in \mathcal{C}$.

Let $P_\alpha = diag(\mathbf{0}, \cdots, \mathbf{0}, \mathbf{1}_{N_\alpha}, \mathbf{0}, \cdots, \mathbf{0})$ whose only nontrivial block is at the α-th site that is the N_α-dimensional identity matrix; if $x \in \mathbb{R}^N$, x_α refers to the N-dimensional vector $P_\alpha x$; and if $U \in \mathcal{C}$, U_α refers to the $N\times N$ matrix $P_\alpha U$. It is directly verified for $U, W \in \mathcal{C}$ that

$$P_\alpha(UW) = (P_\alpha U)(P_\alpha W), \quad P_\alpha(Ux) = (P_\alpha U)(P_\alpha x)$$

and it follows that $P_\alpha(U_1U_2\cdots U_jx) = U_{1\alpha}U_{2\alpha}\cdots U_{j\alpha}x_\alpha$.

2.4. Notation

For a vector $x \in \mathbb{R}^N$, $|x| := \max_{i=1,\cdots,N} |x_i|$. If A is a $N\times N$ matrix, $\|A\|$ denotes the operator norm with respect to the vector norm, i.e., $\|A\| := \max_{|x|\neq 0} \frac{|Ax|}{|x|}$. If $x(t)$ is an orbit, then the primary norm is the sup norm $\|x\|_{L^\infty}$. It is convenient to use the weighted norm to compensate the growth appropriately. For a given real-valued function θ with a fixed constant a,

$$\|x\|_{L^\infty_\theta([c,d])} := \sup_{t\in[c,d]} \left| x(t)\exp\left(-\int_a^t \theta(\eta)d\eta\right)\right|$$

or $\|x\|_\theta$ for shortly if there is no confusion about the domain. $|x(t)|_\theta := \left|x(t)\exp\left(-\int_a^t \theta(\eta)d\eta\right)\right|$ is the weighted length at time t.

For a family of $N\times N$ upper triangular matrices $\{U(t)\}_{t\in\mathbb{R}}$, let $\lambda_i(t)$, $i=1,\cdots,N$ be the diagonal entries of $U(t)$, which are the eigenvalues of $U(t)$. We also define $\overline{\lambda}(t) \triangleq \max_{i=1,\cdots,N} Re\lambda_i(t)$ and $\underline{\lambda}(t) \triangleq \min_{i=1,\cdots,N} Re\lambda_i(t)$. For a block family $\{U_\alpha(t)\}_{t\in\mathbb{R}}$, $\lambda_{\alpha,i}(t)$, $\overline{\lambda}_\alpha(t)$, and $\underline{\lambda}_\alpha(t)$ are similarly defined.

3. Main Results

Suppose $\{U(t)\}_{t\in\mathbb{R}}$ is a family of upper triangular matrices satisfying $(A0)$. In this section, y solves the system

$$y'(t) = U(t)y(t), \tag{3.1}$$

which we call the upper triangular system and x solves the perturbed system

$$x'(t) = U(t)x(t) + \mathcal{E}(t)x(t). \tag{3.2}$$

We denote the family of solution matrices for (3.1) as $\{\Phi(t,\tau)\}_{t,\tau\in\mathbb{R}}$. To study asymptotic stability for (3.2), we need to know that for (3.1), or estimates on $\|\Phi(t,\tau)\|$. These estimates were calculated previously in terms of spectral information [8] and we quote the result below.

Proposition 3.1 ([8]). *Let $\{U(t)\}_{t\in\mathbb{R}}$ be a given family of upper triangular matrices in (3.1) satisfying $(A0)$ and $\{\Phi(t,\tau)\}_{t,\tau\in\mathbb{R}}$ be the corresponding fundamental matrices. Then there is a constant $C_{N,K}>0$ such that for any $a\le b$ and any vector $V\in\mathbb{R}^N$,*

(3.3)
$$\frac{e^{\int_a^b \underline{\lambda}(\eta)d\eta}}{C_{N,K}(1+b-a)^{N-1}}|V| \le |\Phi(b,a)V| \le C_{N,K}(1+b-a)^{N-1}e^{\int_a^b \overline{\lambda}(\eta)d\eta}|V|,$$

where the constant $C_{N,K}$ depends only on N and K.

Now, we consider asymptotic stability for the perturbed system (3.2). Let us expose an invariant subspace point of view. Let $U(t) = diag\big(U_0(t), U_1(t)\big)$ with two upper triangular blocks, and split the phase space into

$\mathbb{R}^N = E_0 \oplus E_1$ for invariant subspaces E_0 and E_1 corresponding to the respective blocks. The following treatment is helpful. We add dimension by one to the phase space by appending the dummy variable t ($t' = 1$ is appended to the system). The fixed point $\mathbf{0}$ of the system extends to the invariant line $M := \{t\text{-axis}\}$, and the asymptotic stability is about the invariant line M. From Proposition 3.1, the range of growth rates in E_j, $j = 1, 2$ can be estimated by their respective eigenvalues. If two ranges have sufficient gap as $t \to \infty$, then segregation of the slower subspace persists under perturbations (see [8, Theorem 1]).

As discussed in [8], the result was only the first half of what would be a parallel statement to Levinson's theorem: Consider a diagonal system with eigenvalues $\lambda_j(t)$, $j = 1, \cdots, N$ in ascending order. Select a $\lambda_k(t)$ that is intermediate. For a diagonal system, the splitting is $\mathbb{R}^N = E_1 \oplus E_2 \oplus \cdots \oplus E_N$ of one dimensional invariant subspaces $E_j := \mathit{Span}\ \mathbf{e}_j$, $j = 1, \cdots, N$ of coordinate basis. Levinson's theorem can single out the E_k that spectrally intervenes. Replacing a diagonal matrix by a block diagonal matrix and the one dimensional E_j by those subspaces corresponding to blocks, the result from [8, Theorem 1] corresponds to a persistent segregation $E^s := E_1 \oplus E_2 \oplus \cdots \oplus E_k$ as a whole. Thus, Theorem 3.2 complements the remaining half of the persistence theory. Analogously to Levinson's theorem, we can single out E_k out of E^s.

THEOREM 3.2. *Let $U(t) = diag(U_0(t), U_1(t), \cdots, U_m(t))$ and for $\alpha = 1, \cdots, m$ $U_\alpha(t)$ is upper triangular with dimension $N_\alpha \times N_\alpha$. Let $N = \sum_{\alpha=1}^{m} N_\alpha$. We make the following assumptions*

1. $\int_{a_0}^{\infty} \|\mathcal{E}(t)\|\, dt < \infty$ *for some a_0.*
2. *There are real-valued functions $\overline{\theta}$ and $\underline{\theta}$ and constants $\delta > 0$ and $A \in \mathbb{R}$ by which the followings hold.*
   (a) *$\{1, \cdots, m\} = J_0 \cup J_1 \cup J_2$. J_0, J_1, J_2 are mutually disjoint and J_1 is non-empty.*
   (b) *$\alpha \in J_0$ implies that for any $t_2 \geq t_1$ and any i, $i = 1, \cdots, N_\alpha$*

$$\int_{t_1}^{t_2} Re\, \lambda_{\alpha,i}(t) - \underline{\theta}(t)\, dt \leq A - (N_\alpha - 1 + \delta) \log(1 + t_2 - t_1).$$

(c) *$\alpha \in J_1$ implies that for any $t_2 \geq t_1$ and any i, $i = 1, \cdots, N_\alpha$*

$$\int_{t_1}^{t_2} Re\, \lambda_{\alpha,i}(t) - \overline{\theta}(t)\, dt \leq A - (N_\alpha - 1 + \delta)\log(1 + t_2 - t_1),$$

$$\int_{t_1}^{t_2} \underline{\theta}(t) - Re\, \lambda_{\alpha,i}(t)\, dt \leq A - (N_\alpha - 1 + \delta)\log(1 + t_2 - t_1).$$

(d) *$\alpha \in J_2$ implies that for any $t_2 \geq t_1$ and any i, $i = 1, \cdots, N_\alpha$*

$$\int_{t_1}^{t_2} \overline{\theta}(t) - Re\, \lambda_{\alpha,i}(t)\, dt \leq A - (N_\alpha - 1 + \delta)\log(1 + t_2 - t_1).$$

Let $N_1 = \sum_{J_1} N_\alpha$. Then there is a constant a and an N_1-dimensional subspace E of $\mathbb{R}^N$ such that $x(a) \in E$ implies $\lim_{t\to\infty} |x(t)|_{\overline{\theta}} = 0$ and $x(a) \in E - \{\mathbf{0}\}$ implies $\limsup_{t\to\infty} |x(t)|_{\underline{\theta}} = \infty$.

Proof. We first show that the assumptions imply estimates on fundamental matrices $\Phi_\alpha = P_\alpha \Phi$ for $\alpha = 1, \cdots, m$, that are also block diagonal.

1. For each $\alpha \in J_0$ and $s \geq \tau$ from Proposition 3.1 and our assumptions.

$$\begin{aligned} &\left|\Phi_\alpha(s,\tau) e^{-\int_\tau^s \underline{\theta}(\eta)\, d\eta} y(\tau)\right| \\ &\leq C_1(1 + s - \tau)^{N_\alpha - 1} \exp\left(\int_\tau^s \overline{\lambda}_\alpha(\eta) - \underline{\theta}(\eta)\, d\eta\right) |y(\tau)| \\ &\leq C_2(1 + s - \tau)^{-\delta} |y(\tau)| \end{aligned} \tag{3.4}$$

and for each $\alpha \in J_1 \cup J_2$ and $s \leq \tau$,

$$\begin{aligned} &\left|\Phi_\alpha(s,\tau) e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta} y(\tau)\right| \\ &\leq C_3(1 + \tau - s)^{N_\alpha - 1} \exp\left(\int_s^\tau -\underline{\lambda}_\alpha(\eta) + \underline{\theta}(\eta)\, d\eta\right) |y(\tau)| \\ &\leq C_4(1 + \tau - s)^{-\delta} |y(\tau)|. \end{aligned} \tag{3.5}$$

2. For each $\alpha \in J_0 \cup J_1$ and $s \geq \tau$,

$$\begin{aligned} &\left|\Phi_\alpha(s,\tau) e^{-\int_\tau^s \overline{\theta}(\eta)\, d\eta} y(\tau)\right| \\ &\leq C_5(1 + s - \tau)^{N_\alpha - 1} \exp\left(\int_\tau^s \overline{\lambda}_\alpha(\eta) - \overline{\theta}(\eta)\, d\eta\right) |y(\tau)| \\ &\leq C_6(1 + s - \tau)^{-\delta} |y(\tau)| \end{aligned} \tag{3.6}$$

and for each $\alpha \in J_2$ and $s \le \tau$,

$$\begin{aligned}
&\left|\Phi_\alpha(s,\tau)e^{\int_s^\tau \overline{\theta}(\eta)\,d\eta}y(\tau)\right| \\
&\le C_7(1+\tau-s)^{N_\alpha-1}\exp\left(\int_s^\tau -\underline{\lambda}_\alpha(\eta)+\overline{\theta}(\eta)\,d\eta\right)|y(\tau)| \\
&\le C_8(1+\tau-s)^{-\delta}|y(\tau)|. \qquad (3.7)
\end{aligned}$$

We set $\Psi_j(t,\tau) := \sum_{\alpha\in J_j}\Phi_\alpha(t,\tau)$, $Q_j := \sum_{\alpha\in J_j} P_\alpha$ for $j=0,1,2$, and $C := \max_k C_k$.

Consider a permutation matrix R such that $RU(t) = diag(B_0(t), B_1(t))$, where $B_0(t) = diag\Big(\{U_\alpha\}_{\alpha\in J_0\cup J_1}\Big)$ and $B_1(t) = diag\Big(\{U_\alpha\}_{\alpha\in J_2}\Big)$. From Theorem A.1 with estimates (3.6)-(3.7), we obtain existence of a constant $a \ge a_0$ and (N_0+N_1)-dimensional subspace E^s such that $x(a)\in E^s$ implies $|x(t)|_{\overline{\theta}} \to 0$ as $t\to\infty$. More specifically, we point out followings.

1. Each of orbit $x(t)$ is a solution of integral equation

$$x(t) = y(t) + \int_a^t (\Psi_0(t,\tau)+\Psi_1(t,\tau))\mathcal{E}(\tau)x(\tau)\,d\tau - \int_t^\infty \Psi_2(t,\tau)\mathcal{E}(\tau)x(\tau)\,d\tau,$$

   where $y(t) = (\Psi_0(t,a)+\Psi_1(t,a))y(a)$ with $y(a)\in(Q_0+Q_1)\mathbb{R}^N$.
2. Technically the constant a has been chosen so that $C\int_a^\infty \|\mathcal{E}(\tau)\|\,d\tau < \frac{1}{2}$.
3. The map $y(a)\mapsto x(a)$ from $(Q_0+Q_1)\mathbb{R}^N$ to $\mathbb{R}^N$ is a linear injection.

N_1-dimensional subspace E of E^s corresponds to that with $y(a) \in Q_1\mathbb{R}^N$. In the below $y_j(t) = Q_j y(t)$ and $x_j(t) = Q_j x(t)$ for $j=0,1,2$. Every such orbit $x(t)$ solves

$$\begin{cases}
x_0(t) = \displaystyle\int_a^t \Psi_0(t,\tau)Q_0\big(\mathcal{E}(\tau)x(\tau)\big)\,d\tau, \\
x_1(t) = y_1(t) + \displaystyle\int_a^t \Psi_1(t,\tau)Q_1\big(\mathcal{E}(\tau)x(\tau)\big)\,d\tau, \quad y_1(t) = \Psi_1(t,a)y_1(a), \\
x_2(t) = -\displaystyle\int_t^\infty \Psi_2(t,\tau)Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\,d\tau
\end{cases} \qquad (3.8)$$

with a $y_1(a)\in Q_1\mathbb{R}^N$.

We claim that $x(a)\in E-\{\mathbf{0}\}$ implies that $|x(t)|_{\underline{\theta}}$ has a lower bound away from 0 for all time $t\ge a$. Pick any $\bar{t}\ge a$ then for any $s\in[a,\bar{t}]$,

$x(s)$ also solves the following integral equations.

$$\begin{cases} x_0(s) = \int_a^s \Psi_0(s,\tau) Q_0\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau, \\ x_1(s) = \Psi_1(s,\bar{t})x_1(\bar{t}) - \int_s^{\bar{t}} \Psi_1(s,\tau) Q_1\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau, \\ x_2(s) = \Psi_2(s,\bar{t})x_2(\bar{t}) - \int_s^{\bar{t}} \Psi_2(s,\tau) Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau. \end{cases} \tag{3.9}$$

To obtain $(3.9)_2$, we multiply $(3.8)_2$ by $\Psi_1(s,t)$ and substitute $\Psi_1(s,t)y_1(t) = y_1(s) = x_1(s) - \int_a^s \Psi_1(s,\tau)\mathcal{E}(\tau)x(\tau)\, d\tau$. To obtain equation $(3.9)_3$, we multiply $(3.8)_2$ by $\Psi_2(s,t)$ then

$$\begin{aligned} \Psi_2(s,t)x_2(t) &= -\int_t^\infty \Psi_2(s,\tau) Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau \\ &= -\int_s^\infty \Psi_2(s,\tau) Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau + \int_s^t \Psi_2(s,\tau) Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau \\ &= x_2(s) + \int_s^t \Psi_2(s,\tau) Q_2\big(\mathcal{E}(\tau)x(\tau)\big)\, d\tau. \end{aligned}$$

Define $w_1(s) := \Psi_1(s,\bar{t})x_1(\bar{t})$, $w_2(s) := \Psi_2(s,\bar{t})x_2(\bar{t})$, and $w(s) := w_1(s) + w_2(s)$. Multiplying both sides of (3.9) by $e^{\int_a^s -\underline{\theta}(\eta)\, d\eta}$,

$$\begin{cases} x_0(s)e^{\int_a^s -\underline{\theta}(\eta)\, d\eta} = \int_a^s \big(\Psi_0(s,\tau)e^{\int_\tau^s -\underline{\theta}(\eta)\, d\eta}\big) Q_0\big(\mathcal{E}(\tau)x(\tau)e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big)\, d\tau, \\ x_1(s)e^{\int_a^s -\underline{\theta}(\eta)\, d\eta} = w_1(s)e^{\int_a^s -\underline{\theta}(\eta)\, d\eta} \\ \qquad\qquad - \int_s^{\bar{t}} \big(\Psi_1(s,\tau)e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta}\big) Q_1\big(\mathcal{E}(\tau)x(\tau)e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big)\, d\tau, \\ x_2(s)e^{\int_a^s -\underline{\theta}(\eta)\, d\eta} = w_2(s)e^{\int_a^s -\underline{\theta}(\eta)\, d\eta} \\ \qquad\qquad - \int_s^{\bar{t}} \big(\Psi_2(s,\tau)e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta}\big) Q_2\big(\mathcal{E}(\tau)x(\tau)e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big)\, d\tau. \end{cases} \tag{3.10}$$

Let $\|\cdot\|_{\underline{\theta}} = \|\cdot\|_{L^\infty_{\underline{\theta}}([a,\bar{t}])}$ in the below. From estimates (3.4)-(3.5) we have

$$\begin{aligned} &\int_a^s \big(\Psi_0(s,\tau)e^{\int_\tau^s -\underline{\theta}(\eta)\, d\eta}\big) Q_0\big(\mathcal{E}(\tau)x(\tau)e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big)\, d\tau \\ &\le \int_a^{\bar{t}} \|\Psi_0(s,\tau)e^{\int_\tau^s -\underline{\theta}(\eta)\, d\eta}\|\, \|\mathcal{E}(\tau)\|\, \|x\|_{\underline{\theta}}\, d\tau \le \frac{1}{2}\|x\|_{\underline{\theta}} \end{aligned}$$

and for $j = 1, 2$

$$\int_s^{\bar{t}} \big(\Psi_j(s,\tau) e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta}\big) Q_1\big(\mathcal{E}(\tau) x(\tau) e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big)\, d\tau,$$
$$\le \int_a^{\bar{t}} \|\Psi_j(s,\tau) e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta}\|\, \|\mathcal{E}(\tau)\|\, \|x\|_{\underline{\theta}}\, d\tau \le \frac{1}{2}\|x\|_{\underline{\theta}}$$

and hence from (3.10)

$$|x(s) - w(s)|_{\underline{\theta}} \le \frac{1}{2}\|x\|_{\underline{\theta}} \quad \text{and thus} \quad \|x\|_{\underline{\theta}} \le 2\|w\|_{\underline{\theta}}.$$

From (3.5),

$$\|w\|_{\underline{\theta}} = \sup_{a \le s \le \bar{t}} \left| \Big(\Psi_1(s,\bar{t}) w_1(\bar{t}) + \Psi_2(s,\bar{t}) w_2(\bar{t})\Big) \exp\left(-\int_a^s \underline{\theta}(\eta) d\eta\right) \right|$$
$$\le C|w(\bar{t})|_{\underline{\theta}} = C|x_1(\bar{t}) + x_2(\bar{t})|_{\underline{\theta}}.$$

Therefore, using $|x_1(s) + x_2(s)|_{\underline{\theta}} \le |x(s)|_{\underline{\theta}} \le \|x\|_{\underline{\theta}}$, for any $a \le s \le \bar{t}$

$$(2C)^{-1}|x_1(s) + x_2(s)|_{\underline{\theta}} \le |x_1(\bar{t}) + x_2(\bar{t})|_{\underline{\theta}}.$$

In particular, it holds that

$$|y_1(a)| = |y_1(a)|_{\underline{\theta}} = |x_1(a)|_{\underline{\theta}} \le |x_1(a) + x_2(a)|_{\underline{\theta}}$$

and thus

$$m := (2C)^{-1}|y_1(a)| \le |x_1(\bar{t}) + x_2(\bar{t})|_{\underline{\theta}}, \tag{3.11}$$

and C is not dependent on $\bar{t}$. This inequality holds for every $\bar{t} \ge a$. The claim follows from observations that $x(a) \in E - \{\mathbf{0}\} \Leftrightarrow y(a) \in Q_1\mathbb{R}^N - \{\mathbf{0}\}\big)$ and $x_1(a) = y_1(a)$.

We finally claim that $x(a) \in E - \{\mathbf{0}\}$ implies that $\limsup_{t\to\infty} |x(t)|_{\underline{\theta}} = \infty$. Suppose not. Then $\|x(t)\|_{L^\infty_{\underline{\theta}}[\bar{a},\infty)}$ is bounded for some $\bar{a}$. By taking $\bar{a}$ larger enough if necessary we can make for $j = 1, 2$ and any $s \ge \bar{a}$

$$\int_s^\infty \left| \big(\Psi_j(s,\tau) e^{\int_s^\tau \underline{\theta}(\eta)\, d\eta}\big) Q_j(\mathcal{E}(\tau) x(\tau) e^{\int_a^\tau -\underline{\theta}(\eta)\, d\eta}\big) \right|\, d\tau \le \frac{m}{2}.$$

Using this in the second and the third equations of (3.10), by triangular inequality

$$|w_1(s) + w_2(s)|_{\underline{\theta}} \ge |x_1(s) + x_2(s)|_{\underline{\theta}} - \frac{m}{2} \ge \frac{m}{2}$$

and from (3.5), $|w_1(s) + w_2(s)|_{\underline{\theta}} \le C(1 + t - s)^{-\delta}|x_1(t) + x_2(t)|_{\underline{\theta}}$ for any $t \ge s \ge \bar{a}$. Therefore

$$\frac{m}{2C}(1 + t - s)^\delta \le |x_1(t) + x_2(t)|_{\underline{\theta}} \quad \text{for any } t \ge s \ge \bar{a}.$$

As $\|x(t)\|_{L^\infty_{\underline{\theta}}[\bar{a},\infty)}$ is bounded, this only can hold if $m = 0$ which is a contradiction. $\square$

4. Discussion

Working with upper triangular matrices offer more flexibility in applications, because not all families of matrices allow continuous Jordan factorization, and even if all the blocks do, transforming the matrix to Jordan form may be unnecessary. Returning to the example in Section 1, taking the 3×3 matrix as a whole is sufficient if we are only to show the stability of the trivial solution. Another possibility is that we may only be interested in one block and not the other, hence we have no need to factorize the latter fine.

The following exposition from a point of view of invariant manifold theory on Levinson's theorem is useful. Levinson's theorem applies to a system $x'(t) = \Lambda(t)x(t)$ for $\Lambda(t)$ diagonal with distinct eigenvalues $\{\lambda_j(t)\}_{j=1}^n$. Let $\mathbf{e}_j$ be the coordinate basis and $< \mathbf{q}_1, \cdots, \mathbf{q}_k >$ denote the vector space spanned by $\mathbf{q}_1, \cdots, \mathbf{q}_k$. For such a diagonal system, each $< \mathbf{e}_j >$ is one dimensional invariant subspace and

$$\mathbb{R}^N = < \mathbf{e}_1 > \oplus < \mathbf{e}_2 > \oplus \cdots \oplus < \mathbf{e}_N >$$

is an invariant splitting of the phase space. Putting $t' = 1$ as a dummy equation extends the fixed point $\mathbf{0}$ of the system to an invariant line $M := \{t\text{-axis}\}$ of the extended system. The splitting becomes that of a tangent bundle along M. With this framework, Levinson's theorem can be viewed as the persistence theorem of the splitting.

Let us consider a system $x'(t) = J(t)x(t)$ for a single $N \times N$ Jordan block $J(t)$, where we first suppose that the shared eigenvalue $\lambda(t)$ has geometric multiplicity 1. Then the invariant subspace structure for the system is

$$< \mathbf{e}_1 > \hookrightarrow < \mathbf{e}_1 > \oplus < \mathbf{e}_2 > \hookrightarrow \;\cdots\; \hookrightarrow < \mathbf{e}_1 > \oplus \cdots \oplus < \mathbf{e}_N > .$$

The coordinate basis $\mathbf{e}_1$ is the only eigenvector of $J(t)$ and $\mathbf{e}_j$ is an eigenvector of $\Big(J(t) - \lambda(t)\Big)^j$ with $\Big(J(t) - \lambda(t)\Big)\mathbf{e}_j \in < \mathbf{e}_1 > \oplus \cdots \oplus < \mathbf{e}_{j-1} >$. A natural question is to find the sufficient smallness conditions on perturbations to retain this cascading invariant structure. This system has no spectral gap, since the eigenvalue is shared, which suggests this problem is essentially of a single Jordan block. Jordan system fundamental matrices are explicit, i.e., we know that the orbits in E_j can at most grow at rate $t^{j-1}e^{\int_a^t \lambda(\eta)\, d\eta}$. [5] and [7] characterized the sufficiency conditions (Equation (6.5) in [2, p.210]) for smallness of perturbation.

Now let $J(t)$ be block diagonal with several Jordan blocks. Its hierarchy of invariant subspaces is clear following the previous discussion. Let E_j^α be the j-th cascaded subspace of the α-th block. Now it makes senses to take the spectral gaps between blocks into account, as well as the perturbation sizes. We see that the underlying theory will be combinatorial. The most complete picture is to keep all those structures persistent, and the problem of finding sufficient conditions has been studied previously [2, Theorem 6.6, Equation (6.29)]. The condition is a combination of spectral gap and perturbation smallness condition. [2] used a different approach, confirming the existence of N independent orbits.

From the discussion, it is seen that the multiple Jordan blocks problem consists of two independent problems as follows.

1. Problem on a single Jordan block for internal cascaded invariant subspaces E_j.
2. Problem on a block diagonal matrix for block-wise invariant subspaces E^α.

This paper is on the second problem where a few or all blocks need not have Jordan form. The critical factor is the the availability of estimates $\|\Phi^\alpha(t,s)\|$ and $\|\Phi^\alpha(t,s)^{-1}\|$ for the block fundamental matrices, which is explicit when a block has Jordan form. From Proposition 3.1, estimates are available solely from eigenvalues for upper triangular blocks (fundamental matrices of Jordan block are also upper triangular).

Appendix A.

The following derivation is a small improvement of [8, Theorem 1] with abstract assumptions. The proof is identical.

THEOREM A.1. *Suppose that $U(t) = diag(B_0(t), B_1(t))$ with $B_0(t)$ and $B_1(t)$ of dimensions $N_0 \times N_0$ and $N_1 \times N_1$ respectively. Let $\{\Phi_{B_0}(t,\tau)\}_{t,\tau\in\mathbb{R}}$ and $\{\Phi_{B_1}(t,\tau)\}_{t,\tau\in\mathbb{R}}$ be their fundamental matrices respectively and $N = N_0 + N_1$. We assume the followings.*

1. $\int_{a_0}^\infty \|\mathcal{E}(t)\| < \infty$ *for some a_0.*
2. *There is a real-valued function θ and a constant $C > 0$ such that*
   (a) *for any $t \geq \tau \geq a_0$*

$$\|\Phi_{B_0}(t,\tau)e^{-\int_\tau^t \theta(\eta)\, d\eta}\| \leq C,$$

and for any $\tau \geq a_0$

$$\lim_{t\to\infty} \|\Phi_{B_0}(t,\tau)e^{-\int_\tau^t \theta(\eta)\,d\eta}\| = 0, \tag{A.1}$$

(b) *for any $t \geq \tau \geq a_0$*

$$\|\Phi_{B_1}(\tau,t)e^{\int_\tau^t \theta(\eta)\,d\eta}\| \leq C$$

Then there is a constant a and an N_0-dimensional subspace E of $\mathbb{R}^N$ such that $x(a) \in E$ implies $\lim_{t\to\infty} |x(t)|_\theta = 0$.

Proof. Let $y(a)$ be a vector such that $P_0 y(a) = y(a)$ and $y(t) = \Phi(t,a)y(a)$. We look for a solution of the following integral equation.

$$x(t) = y(t) + \int_a^t P_0\Phi(t,\tau)\mathcal{E}(\tau)x(\tau)\,d\tau - \int_t^\infty P_1\Phi(t,\tau)\mathcal{E}(\tau)x(\tau)\,d\tau. \tag{A.2}$$

If $x(t)$ exists, that $x(t)$ solves (3.2) follows from that each column of $\Phi(t,\tau)$ solves (3.1). In particular, we have $P_0x(a) = y(a)$. Suppose such an $x(t)$ exists. Multiplying both sides by $e^{-\int_a^t \theta(\eta)\,d\eta}$ and using block matrices calculus,

$$\begin{aligned} x(t)e^{-\int_a^t \theta(\eta)\,d\eta} &= y(t)e^{-\int_a^t \theta(\eta)\,d\eta} \\ &\quad + \int_a^t \left(\Phi_0(t,\tau)e^{-\int_\tau^t \theta(\eta)\,d\eta}\right)\left(\mathcal{E}(\tau)x(\tau)e^{-\int_a^\tau \theta(\eta)\,d\eta}\right)_0 d\tau \\ &\quad - \int_t^\infty \left(\Phi_1(t,\tau)e^{\int_t^\tau \theta(\eta)\,d\eta}\right)\left(\mathcal{E}(\tau)x(\tau)e^{-\int_a^\tau \theta(\eta)\,d\eta}\right)_1 d\tau. \end{aligned} \tag{A.3}$$

Since $\|\mathcal{E}(t)\|$ is integrable, we can choose a so that $C\int_a^\infty \|\mathcal{E}(\tau)\|\,d\tau < \frac{1}{2}$. Then, we have

$$\begin{aligned} |x(t) - y(t)|_\theta &\leq \int_a^t \|\Phi_0(t,\tau)e^{-\int_\tau^t \theta(\eta)\,d\eta}\|\,\|\mathcal{E}(\tau)\|\,|x(\tau)|_\theta\,d\tau \\ &\quad + \int_t^\infty \|\Phi_1(t,\tau)e^{\int_t^\tau \theta(\eta)\,d\eta}\|\,\|\mathcal{E}(\tau)\|\,|x(\tau)|_\theta\,d\tau \\ &\leq \frac{1}{2}\|x\|_\theta. \end{aligned}$$

Let y be fixed and S^y be the operator on $L_\theta^\infty([a,\infty))$ that maps $x \in L_\theta^\infty([a,\infty))$ to the function the right-hand-side of (A.2) defines. The previous estimate shows that $\|S^y x\|_\theta \leq \|y\|_\theta + \frac{1}{2}\|x\|_\theta < \infty$ and thus $S^y x \in L_\theta^\infty([a,\infty))$, hence the solution of the integral equation is the fixed point of the operator. The same estimate shows that $\|S^y x - S^y \bar{x}\|_\theta \leq$

$\frac{1}{2}\|x-\bar{x}\|_\theta$. By the contraction mapping principle, there is a unique fixed point.

It is clear that $\|x\|_\theta \le 2\|y\|_\theta$. The map $y(a) \mapsto x(a)$, where the function x is the unique fixed point of S^y, is a linear map from E_0 $\left(= P_0\mathbb{R}^N\right)$ to $\mathbb{R}^N$. As $P_0 x(a) = y(a)$, the linear map is injective and has rank N_0. E is then the range space.

We know that $|y(t)|_\theta \to 0$ as $t \to \infty$. It remains to show that $|x(t)|_\theta \to 0$ as $t \to \infty$ as well. We show that $|x(t) - y(t)|_\theta \to 0$. Let the first integral in (A.3) be I_1 and the second be I_2. I_2 converges to 0 since

$$\begin{aligned}\lim_{t\to\infty}\int_t^\infty &\|\Phi_1(t,\tau)e^{\int_t^\tau \theta(\eta)\,d\eta}\|\,\|\mathcal{E}(\tau)\|\,|x(\tau)|_\theta\,d\tau\\ &\le 2C\|y\|_\theta \lim_{t\to\infty}\int_t^\infty \|\mathcal{E}(\tau)\|\,d\tau = 0.\end{aligned}$$

For I_1, we divide the integral into

$$\left(\int_a^{t_1} + \int_{t_1}^{t}\right)\left\{\left(\Phi_0(t,\tau)e^{-\int_\tau^t \theta(\eta)\,d\eta}\right)\left(\mathcal{E}(\tau)x(\tau)e^{-\int_a^\tau \theta(\eta)\,d\eta}\right)_0 d\tau\right\}.$$

for some $t_1 \le t$. For any $\epsilon > 0$, we can choose t and t_1 so large enough, while retaining $t \ge t_1$, that $I_1 \le \epsilon$ in the following manner. With the same reasoning used for I_2, we can choose t_1 so large that the integral over $[t_1, t]$ is smaller than $\frac{\epsilon}{2}$. We can express the integral over $[a, t_1]$ as

$$\left(\Phi_0(t,t_1)e^{-\int_{t_1}^t \theta(\eta)\,d\eta}\right)\int_a^{t_1}\left(\Phi_0(t_1,\tau)e^{-\int_\tau^{t_1} \theta(\eta)\,d\eta}\right)\left(\mathcal{E}(\tau)x(\tau)e^{-\int_a^\tau \theta(\eta)\,d\eta}\right)_0 d\tau.$$

From (A.1), $\|\Phi_0(t,t_1)e^{-\int_{t_1}^t \theta(\eta)\,d\eta}\| \to 0$ as $t \to \infty$, and the integral in $[a, t_1]$ must be finite. Therefore we can choose t so large that the above is smaller than $\frac{\epsilon}{2}$.

□

For convenience, we include Levinson's theorem [9] in the following form.

THEOREM A.2 (Levinson's theorem). *Let $x(t) \in \mathbb{R}^N$ and $x'(t) = \big(\Lambda(t) + \mathcal{E}(t)\big)x$, where $\Lambda(t)$ is a diagonal matrix with diagonal entries $\lambda_j(t)$, $j = 1, \cdots, N$ bounded and $\mathcal{E}(t)$ is a matrix such that $\|\mathcal{E}\|$ integrable, i.e., $\int_{a_0}^\infty \|\mathcal{E}(t)\|dt < \infty$ for some a_0. Fix an index k. Suppose we can find some constant A so that either of the following two membership conditions holds for every i.*

$i \in I_1$ if

$$\int_{a_0}^{\infty} Re(\lambda_k(s) - \lambda_i(s))\, ds \to \infty \quad \text{as } t \to \infty \text{ for some } a_0, \tag{A.4}$$

$$\int_{t_1}^{t_2} Re(\lambda_k(s) - \lambda_i(s))\, ds > -A, \quad \text{whenever } t_2 \geq t_1 \geq 0 \tag{A.5}$$

and $i \in I_2$ if

$$\int_{t_1}^{t_2} Re(\lambda_k(s) - \lambda_i(s))\, ds < A, \quad \text{whenever } t_2 \geq t_1 \geq 0. \tag{A.6}$$

Then there is an orbit $\varphi_k(t)$ $t \geq a$ for some a such that,

$$\lim_{t\to\infty} \varphi_k(t) \exp\left(-\int_a^t \lambda_k(s)\, ds\right) = \hat{k}, \tag{A.7}$$

where $\hat{k}$ is the k-th coordinate basis of $\mathbb{R}^d$.

References

[1] S. Bodine and D. A. Lutz, *Asymptotic integration under weak dichotomies*, Rocky Mt. J. Math., **40** (2010), 51-75.

[2] S. Bodine and D. A. Lutz, *Asymptotic Integration of Differential and Difference Equations*, Lecture Notes in Mathematics Vol 2129, Springer International Publishing, Basel, 2015.

[3] K. Chiba and T. Kimura, *On the asymptotic behavior of solutions of a system of linear ordinary differential equations,* Rikkyo Daigaku sugaku zasshi, **18** (1970), 61-80.

[4] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Inc., New York, 1955; pp. 32-58.

[5] W.A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath mathematical monographs, D. C. Heath, 1965.

[6] A. Devinatz and J. L. Kaplan, *Asymptotic estimates for solutions of linear systems of ordinary differential equations having multiple characteristic roots,* Indiana Univ. Math. J., **22** (1972), 355-366.

[7] M.S.P. Eastham, *The Asymptotic Solution of Linear Differential Systems: Application of the Levinson Theorem*, London Mathematical Society Monographs New Series Vol 4, Oxford University Press, Oxford, 1989.

[8] M.-G. Lee, *Asymptotic stability of non-autonomous systems and a generalization of Levinson's theorem*, Mathematics, **7** (2019), 1213.

[9] N. Levinson, *The asymptotic nature of solutions of linear systems of differential equations*, Duke Math. J., **15** (1948), 111-126.

[10] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J., **12** (1960), 305-317.

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